q-analogue of modified KP hierarchy and its quasi-classical limit

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Abstract

A q-analogue of the tau function of the modified KP hierarchy is defined by a change of independent variables. This tau function satisfies a system of bilinear q-difference equations. These bilinear equations are translated to the language of wave functions, which turn out to satisfy a system of linear q-difference equations. These linear q-difference equations are used to formulate the Lax formalism and the description of quasi-classical limit. These results can be generalized to a q-analogue of the Toda hierarchy. The results on the q-analogue of the Toda hierarchy might have an application to the random partition calculus in gauge theories and topological strings.

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1 Introduction

The notion of q-analogues (also called q-deformations) of soliton equations has been studied from a variety of points of view. Kajiwara and Satsuma [1] obtained a q-analogue of the $1+1$ dimensional Toda equation in a bilinear form. Partly motivated by this work, Mironov, Morozov and Vinet [2] introduced a q-analogue of the tau function of the Toda hierarchy. On the other hand, Wu, Zhang and Zheng [3] found a q-analogue of the KdV hierarchy along with soliton solutions. Frenkel [4] addressed this issue from the point of view of W-algebras, and considered a q-analogue of the (generalized) KdV hierarchy and some other related soliton equaitons. Khesin, Lyubashenko and Roger [5] proposed a slightly different framework of q-pseudodifferential operators to formulate the Lax representation of these q-deformed soliton equations. Mas and Seco [6] applied the work of Khesin et al. to a class of q-analogues of W-algebras. Adler, Horozov and van Moerbeke [7] pointed out that these q-analogues are closely related to the ordinary KP and Toda hierarchies. Having obtained a similar result in the context of bispectrality [8], Iliev constructed a q-analogue of the KP hierarchy [9]. Tu [10] studied additional symmetries of Iliev's hierarchy.

In this paper we consider a q-analogue of the modified KP (mKP for short) hierarchy. Our motivation is different from the preceding studies. Namely, we are interested in quasi-classical limit of the q-analogue as the underlying Planck constant $\hbar$ tends to 0. Actually, our true concern lies in the fate of the aforementioned q-analogue of the Toda hierarchy in the quasi-classical limit. The mKP hierarchy, originaly formulated as a system of bilinear equations connecting KP tau functions [11, 12], may be thought of as a subset of the Toda hierarchy [13]. Because of this, one can use the mKP hierarchy as a prototype of technically more complicated consideration on the Toda hierarchy.

In quasi-classical limit, the ordinary Toda hierarchy turns into the dispersionless Toda hierarchy [14]. Accordingly, quasi-classical limit of the mKP hierarchy can be realized as a subsystem of the dispersionless Toda hierarchy. It is natural to expect a similar result for the q-analogue. To address this issue, we need a Lax formalism. We shall construct such a Lax formalism in the same framework as the ordinary Toda hierarchy.

A few comments on the mKP hierarchy are in order. Firstly, various Lax representations other than ours have been proposed for the mKP hierarchy [16, 17, 18]. Secondly, the mKP hierarchy also appears in the work of Adler et al. [7] as the "one-Toda lattice" in their terminology. Thirdly, Takebe [19] studied a generalization of the mKP hierarchy and its dispersionless limit. His construction is based on a different Lax formalism due to Dickey [18].

This paper is organized as follows. Section 2 is a brief review on the tau function of the ordinary mKP hierarchy. In Section 3, its q-analogue is introduced and shown to satisfy a system of bilinear q-difference equations. In Section 4, these bilinear equations are translated to the language of wave functions, which turn out to satisfy a system of linear q-difference equations. These linear q-difference equations are used in Sections 5 and 6 to formulate the Lax formalism and the description of quasi-classical limit. Section 7 is devoted to the case of the q-analogue of the Toda hierarchy.

2 Tau function of modified KP hierarchy

The tau function $\tau(s,t)$ of the mKP hierarchy depends on a discrete ($\mathbf{Z}$ valued) variable s and a set of continuous variables $t = (t_1, t_2, \ldots)$, and satisfies the bilinear equations

$$\oint_{\lambda=\infty} \tau(s', t' - [\lambda^{-1}])\tau(s, t + [\lambda^{-1}])\lambda^{s'-s} e^{\xi(t'-t,\lambda)} d\lambda = 0 \tag{1}$$

for $s' \geq s$ and arbitrary values of t' and t [11, 12]. The contour of the integral is understood to be a circle surrounding $\lambda = \infty$, and the following standard notations are used:

$$[\alpha] = (\alpha, \alpha^2/2, \ldots, \alpha^k/k, \ldots), \quad \xi(t,\lambda) = \sum_{n=1}^{\infty} t_n \lambda^n.$$

The bilinear equation for $s' = s$ coincides with that of the KP hierarchy, so that $\tau(s,t)$ is a tau function of the KP hierarchy. The bilinear equations for $s' \neq s$ define a relation (a Bäcklund transformation in a generalized sense) connecting two solutions of the KP hierarchy. A pair of KP tau functions in such a relation is generated, for instance, by the action of a vertex operator [11, 12]. If the chain of those KP tau functions $\tau(s,t)$ is periodic, i.e., $\tau(s + N, t) = \tau(s,t)$ for a positive integer N, the relation reduces to the ordinary cyclic Darboux transformations for the N-th generalized KdV hierarchy. In the simplest case of $N = 2$, the reduced hierarchy with two tau functions $\tau(0,t)$ and $\tau(1,t)$ is nothing but the modified KdV hierarchy. This explains the origin of the word "modified" in the name of the hierarchy.

Actually, yet another "modified" set of bilinear equations can be derived from these fundamental bilinear equations by shifting t' as

$$t' \to t' - [\lambda_1^{-1}] - \cdots - [\lambda_n^{-1}],$$

$\lambda_1, \ldots, \lambda_n$ being assumed to be on the far side (i.e., in the domain that contains $\lambda = \infty$) of the integration contour. The exponential function $e^{\xi(t'-t,\lambda)}$ thereby gives rise to an extra

factor

$$\prod_{j=1}^{n} \exp\left(-\sum_{k=1}^{\infty} \frac{\lambda^k}{k\lambda_j^k}\right) = \prod_{j=1}^{n}\left(1 - \frac{\lambda}{\lambda_j}\right),$$

so that the bilinear equations take the "second-modified" form

$$\oint_{\lambda=\infty} \tau(s', t' - [\lambda_1^{-1}] - \cdots - [\lambda_n^{-n}] - [\lambda^{-1}])\tau(s, t + [\lambda^{-1}]) \\ \times \lambda^{s'-s} e^{\xi(t'-t,\lambda)}(\lambda - \lambda_1)\cdots(\lambda - \lambda_n)d\lambda = 0. \tag{2}$$

If λ_j's are suitably chosen, as we shall see, these bilinear equations turn into q-difference equations.

3 q-analogue of mKP tau functions

We now introduce a new set of continuous variables $x = (x_1, x_2, \ldots)$ and a set of parameters $q = (q_1, q_2, \ldots)$ with $|q_1| < 1$, $|q_2| < 1$, etc., and consider the multivariate deformation

$$\tau_q(s, t, x) = \tau(s, t + \sum_{n=1}^{\infty} [x_n]_{q_n}^{(n)}) \tag{3}$$

as a q-analogue of the foregoing mKP tau function. In this definition, $[\alpha]_q^{(n)}$ stands for a q-analogue of the $[\alpha]$ symbol of the form

$$[\alpha]_q^{(n)} = \left(0, \ldots, 0, \alpha, 0, \ldots, 0, \frac{(1-q)^2\alpha^2}{2(1-q^2)}, \ldots, 0, \ldots, 0, \frac{(1-q)^k\alpha^k}{k(1-q^k)}, \ldots\right)$$

with the non-zero elements $(1-q)^k\alpha^k/k(1-q^k)$ placed in the kn-th component for $k = 1, 2, \ldots$. This is essentially the same change of variables as proposed by Mironov et al. [2] for the case of the Toda hierarchy, except that we keep the old variables t_1, t_2, ... along with the new ones x_1, x_2, ... If the x_n's other than x_1 are set to zero, $\tau_q(s, t, x)$ reduces to the univariate deformation

$$\tau_q(t, x) = \tau(t + [x]_q), \quad [x]_q = [x]_q^{(1)}, \tag{4}$$

of the KP tau function $\tau(t)$ studied by Adler et al. [7] and Iliev [9] in their work on a q-analogue of the KP hierarchy.

$[\alpha]_q^{(n)}$ is defined so as to satisfy the q-difference relation

$$[q\alpha]_q^{(n)} = [\alpha]_q^{(n)} - [(1-q)\alpha]^{(n)}, \tag{5}$$

where $[\alpha]^{(n)}$ denotes a modification of $[\alpha]$ of the form

$$[\alpha]^{(n)} = \left(0, \ldots, 0, \alpha, 0, \ldots, 0, \frac{\alpha^2}{2}, \ldots, 0, \ldots, 0, \frac{\alpha^k}{k}, \ldots\right)$$

in which the k-th entry α^k/k of $[\alpha]$ is relocated to the kn-th component and the other components are set to zero. This is a generalization of the q-difference relation of $[x]_q$ that lies in the heart of the work of Adler et al. [7] and Iliev [9]. As noted by Mironov et al. [2], $[\alpha]^{(n)}$ can be expressed as a linear combination of $[\alpha]$ symbols,

$$[\alpha]^{(n)} = \sum_{j=1}^{n} [e^{2\pi ij/n}\alpha^{1/n}],$$

so that the q-difference relation of $[\alpha]_q^{(n)}$ can be rewritten as

$$[q\alpha]_q^{(n)} = [\alpha]_q^{(n)} - \sum_{j=1}^{n} [e^{2\pi ij/n}(1-q)^{1/n}\alpha^{1/n}]. \tag{6}$$

This q-difference relation enables us to relate the multiplication $x_n \to q_n x_n$ in $\tau_q(s,t,x)$ to the shift $t \to t - [\lambda_1^{-1}] - \cdots - [\lambda_n^{-1}]$ in (2). If λ_j's are indeed set to the values

$$\lambda_j = e^{-2\pi ij/n}(1-q_n)^{-1/n} x_n^{-1/n}$$

that can be read off from (6), the last polynomial factor in (2) takes the remarkably simple form

$$(\lambda - \lambda_1) \cdots (\lambda - \lambda_n) = \lambda^n - (1-q_n)^{-1} x_n^{-1}.$$

Thus the modified bilinear equation (2) can be converted to the bilinear equation

$$\oint_{\lambda=\infty} \tau_q(s', t' - [\lambda^{-1}], \ldots, q_n x_n, \ldots) \tau_q(s, t + [\lambda^{-1}], \ldots, x_n, \ldots) \\ \times \lambda^{s'-s} e^{\xi(t'-t,\lambda)} (1 - (1-q_n) x_n \lambda^n) d\lambda = 0 \tag{7}$$

for $\tau_q(s,t,x)$. Moreover, iterating (6) k times gives higher order q-difference relations

$$[q_n^k x_n]_{q_n}^{(n)} = [x_n]_{q_n}^{(n)} - \sum_{j=1}^{n} \sum_{\ell=0}^{k-1} [e^{2\pi ij/n}(1-q_n)^{1/n} x_n^{1/n} q_n^{\ell/n}], \tag{8}$$

from which a similar bilinear equation can be derived. Actually, one can consider simultaneous q-multiplication of x_n's as $x_n \to q_n^{k_n} x_n$ (but $k_n = 0$ except for a finite number of n's), which leads to bilinear equations of the more general form

$$\oint_{\lambda=\infty} \tau_q(s', t' - [\lambda^{-1}], x') \tau_q(s, t + [\lambda^{-1}], x) \lambda^{s'-s} e^{\xi(t'-t,\lambda)} \\ \times \prod_{n=1}^{\infty} \prod_{\ell_n=0}^{k_n-1} (1 - (1-q_n) q_n^{\ell_n} x_n \lambda^n) d\lambda = 0, \tag{9}$$

where $x' = (x'_1, x'_2, \ldots)$ and $x = (x_1, x_2, \ldots)$ are related by nonnegative powers $q_1^{k_1}, q_2^{k_2}, \ldots$ of $q_1, q_2, \ldots$ as

$$x'_1 = q_1^{k_1} x_1, \ x'_2 = q_2^{k_2} x_2, \ \ldots. \tag{10}$$

4 q-analogue of mKP Wave functions

Since the time variables t of the mKP hierarchy are retained in the definition of $\tau_q(x, t, x)$, one can define an associated wave function $\Psi_q(s, t, x, \lambda)$ and its conjugate $\Psi_q^*(s, t, x, \lambda)$ as wave functions of the mKP hierarchy:

$$\begin{aligned}
\Psi_q(s, t, x, \lambda) &= \frac{\tau_q(s, t - [\lambda^{-1}], x)}{\tau_q(s, t, x)} \lambda^s e^{\xi(t,\lambda)} e_q(x, \lambda), \\
\Psi_q^*(s, t, x, \lambda) &= \frac{\tau_q(s, t + [\lambda^{-1}], x)}{\tau_q(s, t, x)} \lambda^{-s} e^{-\xi(t,\lambda)} e_q(x, \lambda)^{-1}.
\end{aligned} \tag{11}$$

Note that the exponential part $\lambda^{\pm s} e^{\pm \xi(t,x)}$ is multiplied by the extra factor

$$e_q(x, \lambda) = \prod_{n=1}^{\infty} e_{q_n}^{x_n \lambda^n}, \tag{12}$$

where e_q^z denotes the so called q-exponential function

$$e_q^z = \exp\left(\sum_{k=1}^{\infty} \frac{(1-q)^k z^k}{k(1-q^k)}\right) = \prod_{k=0}^{\infty} (1 - (1-q) z q^k)^{-1}.$$

This extra factor originates in the shift of t by the sum of $[x_n]_{q_n}^{(n)}$ in the definition of $\tau_q(s, t, x)$, and behaves exactly like the ordinary exponential factor under the action of q-difference operators.

A prototype of this q-exponential factor is the q-exponential function $e_q^{x\lambda}$ that plays a central role in the q-analogue of the KP hierarchy [7, 9, 10]. It satisfies the q-difference equation

$$D_q(x) e_q^{x\lambda} = \lambda e_q^{x\lambda} \tag{13}$$

for the q-difference operator $D_q(x)$ that acts on a function of x as

$$D_q(x) f(x) = \frac{f(qx) - f(x)}{qx - x}.$$

Let us also mention that the usual Leibniz rule is modified to this operator as

$$D_q(x)(fg) = D_q(x) f \cdot g + T_q(x) f \cdot D_q(x) g, \tag{14}$$

where $T_q(x)$ denotes the q-multiplication operator

$$T_q(x)f(x) = f(qx).$$

By the same token, as implicitly noted in the work of Mironov et al. [2], the q-exponential factor $e_q(x, \lambda)$ satisfies the q-difference equation

$$D_{q_n}(x_n)e_q(x, \lambda) = \lambda^n e_q(x, \lambda) \tag{15}$$

or, equivalently,

$$T_{q_n}(x_n)e_q(x, \lambda) = (1 - (1 - q_n)x_n\lambda^n)e_q(x, \lambda). \tag{16}$$

Iterating the last equation yields

$$T_{q_n}(x_n)^{k_n}e_q(x, \lambda) = \prod_{\ell_n=0}^{k_n-1}(1 - (1 - q_n)q_n^{\ell_n}x_n\lambda^n) \cdot e_q(x, \lambda).$$

The prefactor on the right hand side coincides with the last polynomial factor in (9). This means that (9) is actually a bilinear equation for the wave functions:

$$\oint_{\lambda=\infty} \Psi_q(s', t', x', \lambda)\Psi_q^*(s, t, x, \lambda)d\lambda = 0. \tag{17}$$

We can now follow a standard procedure to derive a system of linear q-difference equations for $\Psi_q(s, t, x, \lambda)$ from the bilinear equation (17). A technical clue is the following lemma.

Lemma 1 *If $\Phi(s, t, x, \lambda)$ is a function (or Laurent series) of the form*

$$\Phi(s, t, x, \lambda) = \sum_{n=1}^{\infty} \phi_n(s, t, x)\lambda^{-n} \cdot \lambda^s e^{\xi(t,\lambda)}e_q(x, \lambda) \tag{18}$$

and satisfies the bilinear equation

$$\oint_{\lambda=\infty} \Phi(s', t, x, \lambda)\Psi_q^*(s, t, x, \lambda)d\lambda = 0 \tag{19}$$

for $s' \geq s$ and arbitrary values of (t, x), then $\Phi(s, t, x, \lambda) = 0$.

Proof. The bilinear equation in the statement of the lemma implies the equations

$$\sum_{n=1}^{s'-s+1} \phi_n(s', t, x)w^*_{s'-s-n+1}(s, t, x) = 0$$

for $\phi_n(s,t,x)$'s and the coefficients of the Laurent expansion

$$\Psi_q^*(s,t,x,\lambda) = \Big(1 + \sum_{n=1}^{\infty} w_n^*(s,t,x)\lambda^{-n}\Big)\lambda^{-s}e^{-\xi(t,\lambda)}e_q(x,\lambda)^{-1}$$

of $\Psi_q^*(s,t,x,\lambda)$. Starting with the case of $s' = s$ where this equation reduces to $\phi_1(s,t,x) = 0$, one can easily show by induction that $\phi_n(s,t,x) = 0$ for all n. $\square$

To derive a linear q-difference equation for $\Psi_q(s,t,x,\lambda)$, let us note that the action of $D_{q_n}(x_n)$ on $\Psi_q(s,t,x,\lambda)$ yields a function of the form $(\lambda^n + \cdots)\lambda^s e^{\xi(t,\lambda)}e_q(x,\lambda)$, where $\lambda + \cdots$ is a Laurent series of λ that starts from λ^n and continues to terms of $\lambda^{n-1}, \lambda^{n-2}, \ldots$. One can find a difference operator C_n (i.e., a linear combination of the shift operators $e^{m\partial_s}$ that act on a function of s as $e^{m\partial_s}f(s) = f(s+m)$) of the form

$$C_n = e^{n\partial_s} + \sum_{m=1}^{n} c_{nm}(s,t,x)e^{(n-m)\partial_s} \tag{20}$$

that absorbs the main part of the Laurent expansion of $D_{q_n}(x_n)\Psi_q(s,t,x,\lambda)$ as

$$D_{q_n}(x_n)\Psi_q(s,t,x,\lambda) - C_n\Psi_q(s,t,x,\lambda) = O(\lambda^{-1})\lambda^s e^{\xi(t,\lambda)}e_q(x,\lambda).$$

Let $\Phi(s,t,x,\lambda)$ denote this quantity. One can derive, from (17), such relations as

$$\oint_{\lambda=\infty} T_{q_n}(x_n)\Psi_q(s,t,x,\lambda)\cdot\Psi_q^*(s,x,t,\lambda)d\lambda = 0$$

and

$$\oint_{\lambda=\infty} e^{m\partial_s}\Psi_q(s,t,x,\lambda)\cdot\Psi_q^*(s,t,x,\lambda)d\lambda = 0$$

for $m \geq 0$. These relations imply that the assumption of the aforementioned lemma holds for $\Phi(s,t,x,\lambda)$, which thereby turns out to vanishes identically. One can thus confirm that $\Psi_q(s,t,x,\lambda)$ satisfies the linear q-difference equation

$$D_{q_n}(x_n)\Psi_q(s,t,x,\lambda) = C_n\Psi_q(s,t,x,\lambda). \tag{21}$$

5 q-analogue of Lax, Zakharov-Shabat, and Sato equations

Consistency of the linear q-difference equations (21) leads to a system of zero-curvature equations for C_n's. To derive the consistency condition, we start from the commutativity relation

$$D_{q_n}(x_n)D_{q_m}(x_m)\Psi_q(s,t,x,\lambda) = D_{q_m}(x_m)D_{q_n}(x_n)\Psi_q(s,t,x,\lambda),$$

and calculate both hand sides using (21) and the q-difference Leibniz rule. The outcome is the equation

$$\Big(D_{q_n}(x_n)C_m - D_{q_m}(x_m)C_n + T_{q_n}(x_n)C_m \cdot C_n - T_{q_m}(x_m)C_n \cdot C_m\Big)\Psi_q(s,t,x,\lambda) = 0,$$

which implies that the q-difference analogue

$$D_{q_n}(x_n)C_m - D_{q_m}(x_m)C_n + T_{q_n}(x_n)C_m \cdot C_n - T_{q_m}(x_m)C_n \cdot C_m = 0 \tag{22}$$

of the zero-curvature equation (also called "Zakharov-Shabat equations") is satisfied.

We now introduce the dressing operator (also called "Sato-Wilson operator")

$$W = 1 + \sum_{n=1}^{\infty} w_n(s,t,x)e^{-n\partial_s}, \tag{23}$$

where $w_n(s,t,x)$ denote the coefficients of the Laurent expansion

$$\Psi_q(s,t,x,\lambda) = \Big(1 + \sum_{n=1}^{\infty} w_n(s,t,x)\lambda^{-n}\Big)\lambda^s e^{\xi(t,\lambda)} e_q(x,\lambda).$$

(21) can be converted to the q-difference equations

$$D_{q_n}(x_n)W = C_nW - T_{q_n}(x_n)W \cdot e^{n\partial_s}. \tag{24}$$

One can eliminate C_n to derive a nonlinear system of equations for W. This is achieved by multiplying both hand sides by W from the right side,

$$D_{q_n}(x_n)W \cdot W^{-1} = C_n - T_{q_n}(x_n)W \cdot e^{n\partial_s}W^{-1},$$

and separating the difference operators into the part of nonnegative/negative powers of e^{∂_s}. Let $(A)_{\geq 0}$ and $(A)_{<0}$ denote the projection of a difference operator to these parts:

$$\Big(\sum_n a_n e^{n\partial_s}\Big)_{\geq 0} = \sum_{n \geq 0} a_n e^{n\partial_s}, \quad \Big(\sum_n a_n e^{n\partial_s}\Big)_{<0} = \sum_{n<0} a_n e^{n\partial_s}.$$

The $(\cdot)_{\geq 0}$ part of the foregoing q-difference equation gives the relation

$$C_n = \Big(T_{q_n}(x_n)W \cdot e^{n\partial_s}W^{-1}\Big)_{\geq 0}. \tag{25}$$

This enables one to eliminate C_n from the q-difference equation itself. The outcome is the nonlinear q-difference equation

$$D_{q_n}(x_n)W = -\Big(T_{q_n}(x_n)W \cdot e^{n\partial_s}W^{-1}\Big)_{<0}W. \tag{26}$$

This is a q-difference analogue of the so called "Sato equations".

Finally we introduce the Lax operator

$$L = We^{\partial_s}W^{-1} = e^{\partial_s} + \sum_{n=1}^{\infty} u_n e^{(1-n)\partial_s}, \tag{27}$$

which turns out to satisfy the q-difference Lax equations

$$D_{q_n}(x_n)L = C_n L - T_{q_n}(x_n)L \cdot C_n \tag{28}$$

as a consequence of the Sato equations (26). The same Lax equations can also be derived from the consistency of the formal spectral equation

$$L\Psi_q(s,t,x,\lambda) = \lambda\Psi_q(s,t,x,\lambda) \tag{29}$$

with the linear q-difference equations (21). The commutativity, $D_{q_n}(x_n)D_{q_m}(x_m)L = D_{q_m}(x_m)L$, of the flows of the q-difference Lax equations is ensured by the q-difference Zakharov-Shabat equations (22).

To conclude, let us mention that $\Psi_q(s,t,x,\lambda)$ satisfies the linear equations

$$\partial_{t_n}\Psi_q(s,t,x,\lambda) = B_n\Psi_q(s,t,x,\lambda) \tag{30}$$

of the ordinary mKP hierarchy as well, B_n being a difference operator of the form

$$B_n = e^{n\partial_s} + \sum_{m=1}^{n} b_{nm}(s,t,x)e^{(n-m)\partial_s}. \tag{31}$$

One can derive from these linear equations the Sato equations

$$\partial_{t_n}W = -\left(We^{n\partial_s}W^{-1}\right)_{<0}W, \tag{32}$$

the Zakharov-Shabat equations

$$\partial_{t_n}B_m - \partial_{t_m}B_n + [B_m, B_n] = 0, \tag{33}$$

and the Lax equations

$$\partial_{t_n}L = [B_n, L] \tag{34}$$

along with the relation

$$B_n = \left(We^{n\partial_s}W^{-1}\right)_{\geq 0} = (L^n)_{\geq 0} \tag{35}$$

connecting B_n with W and L. These equations comprise a Lax formalism of the mKP hierarchy. What we have derived above is a q-difference analogue thereof.

6 Quasi-classical limit and Hamilton-Jacobi equations

We now turn to the issue of quasi-classical limit. To this end, we set the parameters q_n to depend on the Planck constant $\hbar$ as

$$q_n = q^n = e^{-n\beta\hbar}, \quad q = e^{-\beta\hbar}, \tag{36}$$

where β is an arbitrary constant, and rescale the independent variables s, t_n, x_n as

$$s \to s/\hbar, \quad t_n \to t_n/\hbar, \quad x_n \to x_n/(1-q^n). \tag{37}$$

The fundamental operators $e^{m\partial_s}$, ∂_{t_n} and $D_{q^n}(x_n)$ are thereby rescaled as

$$e^{m\partial_s} \to e^{m\hbar\partial_s}, \quad \partial_{t_n} \to \hbar\partial_{t_n}, \quad D_{q^n}(x_n) \to (1-q^n)D_{q^n}(x_n),$$

which resembles the usual set-up of quantum mechanics. The linear equations of the wave functions are reformulated in the $\hbar$-dependent form as

$$\hbar\partial_{t_n}\Psi_q(\lambda) \quad = \quad B_n\Psi_q(\lambda) \tag{38}$$

and

$$(1-q^n)D_{q^n}(x_n)\Psi_q(\lambda) \quad = \quad C_n\Psi_q(\lambda). \tag{39}$$

Also note that we have slightly changed the notation, namely, we omit writing the dependence on $x, t, x, \hbar$ to highlight the dependence on λ. The coefficients b_{nm} and c_{nm} of

$$B_n = e^{n\hbar\partial_s} + \sum_{m=1}^{n} b_{nm}e^{(n-m)\hbar\partial_s}, \quad C_n = e^{n\hbar\partial_s} + \sum_{m=1}^{n} c_{nm}e^{(n-m)\hbar\partial_s},$$

too, are understood to depend on $\hbar$ arbitrarily. To achieve reasonable quasi-classical limit, however, we assume that b_{nm} and c_{nm} have a smooth limit

$$\lim_{\hbar\to 0} b_{nm} = b^0_{nm}, \quad \lim_{\hbar\to 0} c_{nm} = c^0_{nm} \tag{40}$$

as $\hbar \to 0$. As in the case of the Toda hierarchy [14], this imposes a rather strong condition on the solution in consideration; let us simply assume that these conditions are satisfied. One can readily see from (25) and (35) that B_n and C_n coincide in the limit as $\hbar \to 0$, namely,

$$b^0_{nm} = c^0_{nm}. \tag{41}$$

Under these assumptions, one can set the wave function in the WKB form

$$\Psi_q(\lambda) = \exp\Big(\hbar^{-1}S(\lambda) + O(\hbar^0)\Big) \tag{42}$$

or, more loosely,

$$\Psi_q(\lambda) \sim e^{\hbar^{-1}S(\lambda)}.$$

Since the q-exponential factor $e_q(x,\lambda)$ behaves as

$$e_q(x,\lambda) = \exp\left(\sum_{n=1}^{\infty} \frac{\mathrm{Li}_2(x_n\lambda^n)}{n\beta\hbar} + O(\hbar^0)\right), \tag{43}$$

where $\mathrm{Li}_2(z)$ denotes the dilogarithmic function

$$\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2},$$

the phase function $S(\lambda) = S(s,t,x,\lambda)$ is a Laurent series of the form

$$S(\lambda) = s\log\lambda + \xi(t,\lambda) + \sum_{n=1}^{\infty} \frac{\mathrm{Li}_2(x_n\lambda^n)}{n\beta} + (\text{negative powers of } \lambda). \tag{44}$$

One can derive from (38) and (39) a system of Hamilton-Jacobi equations as follows.

For warm-up, let us first consider (38). Actually, this has been done in the study of quasi-classical limit of the Toda hierarchy [14]. Upon substituting the WKB ansatz, both hand sides of (38) can be expressed as

$$\partial_{t_n}\Psi_q(\lambda) \sim \partial_{t_n}S(\lambda)e^{\hbar^{-1}S(\lambda)}, \quad B_n\Psi_q(\lambda) \sim \mathcal{B}_n(e^{\partial_s S(\lambda)})e^{\hbar^{-1}S(\lambda)},$$

where $\mathcal{B}_n(p)$ is a polynomial of the form

$$\mathcal{B}_n(p) = p^n + \sum_{m=1}^{n} b_{nm}^0 p^{n-m}. \tag{45}$$

Thus one obtains the Hamilton-Jacobi equation

$$\partial_{t_n}S(\lambda) = \mathcal{B}_n(e^{\partial_s S(\lambda)}). \tag{46}$$

Let us now consider (39). The main task is to determine the asymptotic form of the left hand side, i.e.,

$$\begin{aligned}
&(1-q^n)D_{x_n}(q^n)\Psi_s(\lambda) \\
&= x_n^{-1}\Big(\exp(\hbar^{-1}S(\lambda) + O(\hbar^0)) - \exp(\hbar^{-1}T_{x_n}(q^n)S(\lambda) + O(\hbar^0))\Big).
\end{aligned}$$

$T_{x_n}(q^n)S(\lambda)$ can be expanded to a Taylor series of $\hbar$ as

$$\begin{aligned} T_{x_n}(q^n)S(s,t,x,\lambda) &= S(s,t,\ldots,e^{-n\beta\hbar}x_n,\ldots,\lambda) \\ &= S(s,t,x,\lambda) - n\beta\hbar x_n\partial_{x_n}S(s,t,x,\lambda) + O(\hbar^2). \end{aligned}$$

Consequently,

$$(1-q^n)D_{x_n}(q^n)\Psi_s(\lambda) \sim x_n^{-1}\Big(1-\exp(-n\beta x_n\partial_{x_n}S(\lambda))\Big)e^{\hbar^{-1}S(\lambda)}.$$

On the other hand, since c_{nm} and b_{nm} coincide in the limit as $\hbar \to 0$,

$$C_n\Psi_q(\lambda) \sim \mathcal{B}_n(e^{\partial_s S(\lambda)})e^{\hbar^{-1}S(\lambda)}.$$

Thus one obtains the Hamilton-Jacobi equation

$$x_n^{-1}\Big(1-\exp(-n\beta x_n\partial_{x_n}S(\lambda))\Big) = \mathcal{B}_n(e^{\partial_s S(\lambda)}). \tag{47}$$

It will be also suggestive to rewrite this equation as

$$x_n\partial_{x_n}S(\lambda) = -\frac{1}{n\beta}\log(1-x_n\mathcal{B}_n(e^{\partial_s S(\lambda)})). \tag{48}$$

One can now follow the case of the Toda hierarchy [14] to convert these Hamilton-Jacobi equations to a Lax formalism of the “dispersionless” type [15]. The Lax function $\mathcal{L}(p)$ is obtained by solving the equation

$$\partial_s S(\lambda) = p \tag{49}$$

for λ as

$$\lambda = \mathcal{L}(p) = p + \sum_{n=1}^{\infty} u_n^0 p^{1-n}. \tag{50}$$

The coefficients u_n^0 coincide with the limit of the coefficients u_n of L as $\hbar \to 0$. (46) and (48) correspond to the Lax equations

$$\begin{aligned} \partial_{t_n}\mathcal{L}(p) &= \{\mathcal{B}_n(p), \mathcal{L}(p)\}, \\ x_n\partial_{x_n}\mathcal{L}(p) &= -\frac{1}{n\beta}\{\log(1-x_n\mathcal{B}_n(p)), \mathcal{L}(p)\}, \end{aligned} \tag{51}$$

with respect to the Poisson bracket

$$\{f,g\} = p\partial_p f\cdot\partial_s g - \partial_s f\cdot p\partial_p g.$$

One can further introduce the Orlov-Schulman function

$$\mathcal{M}(p) = \lambda\partial_\lambda S(\lambda)|_{\lambda=\mathcal{L}(p)}, \tag{52}$$

which turns out to satisfy the Lax equations

$$\begin{aligned} \partial_{t_n}\mathcal{M}(p) &= \{\mathcal{B}_n(p), \mathcal{M}(p)\}, \\ x_n\partial_{x_n}\mathcal{M}(p) &= -\frac{1}{n\beta}\{\log(1 - x_n\mathcal{B}_n(p)), \mathcal{M}(p)\} \end{aligned} \tag{53}$$

and the Poisson commutation relation

$$\{\mathcal{L}(p), \mathcal{M}(p)\} = \mathcal{L}(p). \tag{54}$$

7 q-analogue of Toda hierarchy

The tau function $\tau(s, t, \bar{t})$ of the Toda hierarchy [13] depends on yet another set of continuous variables $\bar{t} = (\bar{t}_1, \bar{t}_2, \ldots)$, and satisfy the bilinear equations

$$\begin{aligned} &\oint_{\lambda=\infty} \tau(s', t' - [\lambda^{-1}], \bar{t}')\tau(s, t + [\lambda^{-1}], \bar{t})\lambda^{s'-s}e^{\xi(t'-t,\lambda)}d\lambda \\ &= \oint_{\lambda=0} \tau(s', t', \bar{t}' - [\lambda])\tau(s, t, \bar{t} + [\lambda])\lambda^{s'-s}e^{\xi(\bar{t}'-\bar{t},\lambda^{-1})}d\lambda, \end{aligned} \tag{55}$$

where s' and s ar now aribrary (namely, there is no ordering constraint), and the contours of the integrals on both hand sides are understood to be a circle surrounding $\lambda = \infty$ and $\lambda = 0$ respectively. Following Mironov, Morozov and Vinet [2], we now introduce two sets of continuous variables $x = (x_1, x_2, \ldots)$, $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots)$ along with parameters $q = (q_1, q_2, \ldots)$, $\bar{q} = (\bar{q}_1, \bar{q}_2, \ldots)$, and consider the q-analogue

$$\tau_{q,\bar{q}}(s, t, \bar{t}, x, \bar{x}) = \tau(s,\ t + \sum_{n=1}^{\infty}[x_n]_{q_n}^{(n)},\ \bar{t} + \sum_{n=1}^{\infty}[\bar{x}_n]_{\bar{q}_n}^{(n)}) \tag{56}$$

of $\tau(s, t, \bar{t})$. The modified bilinear equations (2) can be generalized to $\tau_{q,\bar{q}}(s, t, \bar{t}, x, \bar{x})$. One can rewrite those modified bilinear equations for $\tau_{q,\bar{q}}(s, t, \bar{t}, x, \bar{x})$ to the bilinear equations

$$\begin{aligned} &\oint_{\lambda=\infty} \Psi_{q,\bar{q}}(s', t', \bar{t}', x', \bar{x}', \lambda)\Psi^*_{q,\bar{q}}(s, t, \bar{t}, x, \bar{x}, \lambda)d\lambda \\ &= \oint_{\lambda=0} \bar{\Psi}_{q,\bar{q}}(s', t', \bar{t}', x', \bar{x}', \lambda)\bar{\Psi}^*_{q,\bar{q}}(s, t, \bar{t}, x, \bar{x}, \lambda)d\lambda \end{aligned} \tag{57}$$

for the wave functions

$$\Psi_{q,\bar{q}}(s, t, \bar{t}, x, \bar{x}, \lambda) = \frac{\tau_{q,\bar{q}}(s, t - [\lambda^{-1}], \bar{t}, x, \bar{x})}{\tau_{q,\bar{q}}(s, t, \bar{t}, x, \bar{x})}\lambda^s e^{\xi(t,\lambda)}e_q(x, \lambda),$$

$$
\begin{aligned}
\Psi^*_{q,\bar q}(s,t,\bar t,x,\bar x,\lambda) &= \frac{\tau_{q,\bar q}(s,t+[\lambda^{-1}],\bar t,x,\bar x)}{\tau_{q,\bar q}(s,t,\bar t,x,\bar x)}\lambda^{-s}e^{-\xi(t,\lambda)}e_q(x,\lambda)^{-1},\\
\bar\Psi_{q,\bar q}(s,t,\bar t,x,\bar x,\lambda) &= \frac{\tau_{q,\bar q}(s+1,t,\bar t-[\lambda],x,\bar x)}{\tau_{q,\bar q}(s,t,\bar t,x,\bar x)}\lambda^{s}e^{\xi(\bar t,\lambda^{-1})}e_{\bar q}(\bar x,\lambda^{-1}),\\
\bar\Psi^*_{q,\bar q}(s,t,\bar t,x,\bar x,\lambda) &= \frac{\tau_{q,\bar q}(s-1,t,\bar t+[\lambda],x,\bar x)}{\tau_{q,\bar q}(s,t,\bar t,x,\bar x)}\lambda^{-s}e^{-\xi(\bar t,\lambda^{-1})}e_{\bar q}(\bar x,\lambda^{-1})^{-1}.
\end{aligned}
\tag{58}
$$

$x'_n, \bar x'_n, x_n, \bar x_n$ are understood to be related by nonnegative powers $q_n^{k_n}$ and $\bar q_n^{\bar k_n}$ of q_n and $\bar q_n$ as

$$
x'_1 = q_1^{k_1}x_1,\ x'_2 = q_1^{k_2}x_2,\ \ldots,\quad \bar x'_1 = \bar q_1^{\bar k_1}\bar x_1,\ \bar x'_2 = \bar q_2^{\bar k_2}\bar x_2,\ \ldots. \tag{59}
$$

The wave functions turn out to satisfy a set of linear differential and q-difference equations. To this end, the previous lemma has to be generalized to an equation of the form

$$
\begin{aligned}
&\oint_{\lambda=\infty}\Phi(s',t,\bar t,x,\bar x,\lambda)\Psi_{q,\bar q}(s,t,\bar t,x,\bar x,\lambda)d\lambda\\
&\quad = \oint_{\lambda=0}\bar\Phi(s',t,\bar t,x,\bar x,\lambda)\bar\Psi_{q,\bar q}(s,t,\bar t,x,\bar x,\lambda)d\lambda,
\end{aligned}
\tag{60}
$$

where $\Phi(s,t,\bar t,x,\bar x,\lambda)$ is the same as in the previous case and $\bar\Phi(s,t,\bar t,x,\bar x,\lambda)$ is a function of the form

$$
\bar\Phi(s,t,\bar t,x,\bar x,\lambda) = \sum_{n=1}^{\infty}\bar\phi_n(s,t,\bar t,x,\bar x)\lambda^n\cdot\lambda^s e^{\xi(\bar t,\lambda^{-1})}e_{\bar q}(\bar x,\lambda^{-1}). \tag{61}
$$

The generalized statement is that such a pair of functions $\Phi(s,t,\bar t,x,\bar t,\lambda)$ and $\bar\Phi(s,t,\bar t,x,\bar t,\lambda)$ vanish identically. One can threby show that $\Psi_{q,\bar q}(s,t,\bar t,x,\bar x,\lambda)$ satisfy the linear q-difference equations

$$
\begin{aligned}
D_{q_n}(x_n)\Psi_{q,\bar q}(s,t,\bar t,x,\bar x,\lambda) &= C_n\Psi_{q,\bar q}(s,t,\bar t,x,\bar x,\lambda),\\
D_{\bar q_n}(\bar x_n)\Psi_{q,\bar q}(s,t,\bar t,x,\bar x,\lambda) &= \bar C_n\Psi_{q,\bar q}(s,t,\bar t,x,\bar x,\lambda)
\end{aligned}
\tag{62}
$$

along with the linear differential equations

$$
\begin{aligned}
\partial_{t_n}\Psi_{q,\bar q}(s,t,\bar t,x,\bar x,\lambda) &= B_n\Psi_{q,\bar q}(s,t,\bar t,x,\bar x,\lambda),\\
\partial_{\bar t_n}\Psi_{q,\bar q}(s,t,\bar t,x,\bar x,\lambda) &= \bar B_n\Psi_{q,\bar q}(s,t,\bar t,x,\bar x,\lambda)
\end{aligned}
\tag{63}
$$

of the ordinary Toda hierarchy, and that the same equations hold for $\bar\Psi_{q,\bar q}(s,t,\bar t,x,\bar x,\lambda)$ as well. B_n and C_n are the same difference operators as in the previously case. $\bar B_n$ and $\bar C_n$ are difference operators of the form

$$
\bar B_n = \sum_{m=0}^{n-1}\bar b_{nm}e^{(m-n)\partial_s},\quad \bar C_n = \sum_{m=0}^{n-1}\bar c_{nm}e^{(m-n)\partial_s}. \tag{64}
$$

Quasi-classical limit is achieved by setting $q_n = \bar{q}_n = q^n$, $q = e^{-\beta\hbar}$, and rescaling the independent variables as $s \to s/\hbar$, $t_n \to t_n/\hbar$, $\bar{t}_n \to \bar{t}_n/\hbar$, $x_n \to x_n/(1-q^n)$, $\bar{x}_n \to \bar{x}_n/(1-q^n)$. Also the coefficients of $B_n, \bar{B}_n, C_n, \bar{C}_n$ are allowed to depend on $\hbar$ arbitrarily, except that they have a smooth limit as $\hbar \to 0$. Assuming the WKB ansatz

$$\Psi_{qq}(s,\lambda) \sim \exp\Big(\hbar^{-1} S(\lambda) + O(\hbar^0)\Big), \tag{65}$$

one can derive the Hamilton-Jacobi equations

$$\begin{aligned}
\partial_{t_n} S(\lambda) &= \mathcal{B}_n(e^{\partial_s S(\lambda)}), \\
\partial_{\bar{t}_n} S(\lambda) &= \bar{\mathcal{B}}_n(e^{\partial_s S(\lambda)}), \\
x_n \partial_{x_n} S(\lambda) &= -\frac{1}{n\beta} \log(1 - \mathcal{B}_n(e^{\partial_s S(\lambda)})), \\
\bar{x}_n \partial_{\bar{x}_n} S(\lambda) &= -\frac{1}{n\beta} \log(1 - \bar{\mathcal{B}}_n(e^{\partial_s S(\lambda)})),
\end{aligned} \tag{66}$$

where $\mathcal{B}_n(p)$ is the same polynomial as in the case of the modified KP hierarchy and $\bar{\mathcal{B}}_n(p)$ is a polynomial in p^{-1} of the form

$$\bar{\mathcal{B}}_n(p) = \sum_{m=0}^{n-1} \bar{b}^0_{nm} p^{m-n}, \quad \bar{b}^0_{nm} = \lim_{\hbar \to 0} \bar{b}_{nm}. \tag{67}$$

One can define the Lax function $\mathcal{L}(p)$ and the Orlov-Schulman function $\mathcal{M}(p)$ in the same way as the previous case. They satisfy the Poisson commutation relation

$$\{\mathcal{L}(p), \mathcal{M}(p)\} = \mathcal{L}(p) \tag{68}$$

and a system of Lax equations with respect to the same Poisson bracket $\{\cdot,\cdot\}$. The Lax equations for $\mathcal{L}(p)$ read

$$\begin{aligned}
\partial_{t_n} \mathcal{L}(p) &= \{\mathcal{B}_n(p), \mathcal{L}(p)\}, \\
\partial_{\bar{t}_n} \mathcal{L}(p) &= \{\bar{\mathcal{B}}_n(p), \mathcal{L}(p)\}, \\
x_n \partial_{x_n} \mathcal{L}(p) &= -\frac{1}{n\beta} \{\log(1 - x_n \mathcal{B}_n(p)), \mathcal{L}(p)\}, \\
\bar{x}_n \partial_{\bar{x}_n} \mathcal{L}(p) &= -\frac{1}{n\beta} \{\log(1 - \bar{x}_n \bar{\mathcal{B}}_n(p)), \mathcal{L}(p)\},
\end{aligned} \tag{69}$$

and Lax equations of the same form hold for $\mathcal{M}(p)$. One can repeat the same calculations for the WKB ansatz

$$\bar{\Psi}_{q,\bar{q}}(\lambda) \sim \exp\Big(\hbar^{-1} \bar{S}(\lambda) + O(\hbar^0)\Big) \tag{70}$$

of $\bar{\Psi}_{q,\bar{q}}(\lambda)$. This eventually leads to the second pair $\bar{\mathcal{L}}(p), \bar{\mathcal{M}}(p)$ of Lax and Orlov-Schulman functions. Using the four functions $\mathcal{L}(p), \mathcal{M}(p), \bar{\mathcal{L}}(p), \bar{\mathcal{M}}(p)$, one can formulate a "twistor theory" [15] of the extended Lax formalism.

8 Conclusion

We have considered a q-analogue of the mKP hierarchy as a prototype of the q-analogue of the Toda hierarchy. As expected, we have been able to describe the quasi-classical limit of both systems by the method developed for the Toda hierarchy [14]. An unusual feature is that the q-difference flows turn into the flows with logarithmic generators such as $\log(1 - x_n\mathcal{B}_n(p))$ and $\log(1 - \bar{x}_n\bar{\mathcal{B}}_n(p))$. This is reminiscent of a logarithmic structure that can be seen in Takebe's result [19].

As a final remark, let us mention a possible link of the present work with the random partition calculus in gauge theories and topological strings [20, 21]. Such a link can be seen in the simple (almost trivial) tau function

$$\tau(s, t, \bar{t}) = \Lambda^{s^2} \exp\left(-\sum_{n=1}^{\infty} \Lambda^{2n} n t_n \bar{t}_n\right) \tag{71}$$

of the Toda hierarchy, where Λ is an arbitrary constant. If $q_n, \bar{q}_n$ are set to the special value q^n, and all $t_n, \bar{t}_n, x_n, \bar{x}_n$ but x_1 and $\bar{x_1}$ are turned off, then the q-deformed tau function reduces to

$$\tau_{q,\bar{q}}(s, x_1, \bar{x}_1) = \Lambda^{s^2} \exp\left(-\sum_{k=1}^{\infty} \frac{(\Lambda^2(1-q)^2 x_1\bar{x}_1)^k}{k(1-q^k)^2}\right), \tag{72}$$

which, upon suitably adjusting the variables $x_1, \bar{x}_1$, coincides with a relevant physical quantity (such as the partition function) in the random partition calculus.

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